Laser: Theory and Modern Application
ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Exercise No.12: Nonlinear frequency conversion - I

12.1 Coherence length for phase matching

Calculate the coherence length for a KDP crystal at $\lambda = 1\mu m$.
The refractive indices are:
$n(2\omega) = 1.514928$
$n(\omega) = 1.496044$
Explain what this coherence length means.

12.2 Angular sensitivity of phase matching

In angle phase matching, we are interested in the angular sensitivity around the angle that provides perfect phase matching.
Consider the case of second harmonic generation. Calculate the phase mismatch $\Delta\kappa L/2$ as a function of the angle mismatch $(\Theta - \Theta_m)$, where Θ_m is the phase matching angle of the birefrigent crystal.
Hint: expand $n_e^{2\omega}(\Theta)$ as a Taylor expansion near Θ_m and keep the first two terms.

12.3 Angle phase matching in non-linear crystals

Calculate the phase matching angle required to most efficiently convert red light (wavelength in vacuum $660nm$) from a high power laser diode to the second harmonic $(330nm)$.
Use a BBO crystal. The ordinary and extraordinary index of refraction are:
$n_o(600nm) = 1.667$
$n_o(300nm) = 1.729$
$n_e(600nm) = 1.549$
$n_e(300nm) = 1.592$

12.4 Coupled-mode equations for four-wave mixing[1]

The evolution of electromagnetic field, propagating through nonlinear medium, is often convenient to describe in terms of the so-called coupled-mode equations. In this exercise you need to derive such equations for light propagation in a waveguide with Kerr nonlinearity.

In presence of nonlinearity the dielectric medium polarization $\vec{P}$ can be separated in linear ($\vec{P}_{\rm L}$) and non-linear ($\vec{P}_{\rm NL}$) parts.

$$\vec{P} = \vec{P}_{\rm L} + \vec{P}_{\rm NL}. \tag{1}$$

The linear part is related to the refractive index n of the material

$$\vec{P}_{\rm L} = \epsilon_0 \chi^{(1)} \vec{E} = \epsilon_0 (n^2 - 1)\vec{E}, \tag{2}$$

and the non-linear part in the case of Kerr nonlinearity in an isotropic non-dispersive dielectric is

$$\vec{P}_{\rm NL} = \epsilon_0 \chi^{(3)} |\vec{E}|^2 \vec{E}, \tag{3}$$

[1]Stolen, R., Bjorkholm, J., 1982. Parametric amplification and frequency conversion in optical fibers. IEEE Journal of Quantum Electronics 18, 1062–1072. https://doi.org/10.1109/JQE.1982.1071660

where $\chi^{(3)}$ is the nonlinear coefficient.

Incorporating the Eq. (1-3) into the Maxwell's equations results in the wave equation for Kerr-nonlinear media[2]

$$\vec{\nabla}^2 \vec{E} - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = \frac{\chi^{(3)}}{c^2}\frac{\partial^2}{\partial t^2}|\vec{E}|^2\vec{E}. \tag{4}$$

The electromagnetic field in our problem is confined to a waveguide, supporting in general multiple propagating modes, and oriented along the z direction of the coordinate system. The nonlinear polarization in the vast majority of practical cases modify the electromagnetic field only at the time scale slow compare to its oscillation period. Thus we can use the waveguide modes $\vec{\psi}_l(x,y)$ as the basis for our analysis of the nonlinear effects. The electric field $\vec{E}$ at the frequency ω can be expanded as

$$\vec{E}(x,y,z,t) = \sum_l A_l \vec{\psi}_l(x,y)\cos(k_l z - \omega t + \phi_l), \tag{5}$$

where l is the guided mode index, A_l, ϕ_l and k_l are the mode amplitudes, phases and propagation constants correspondingly. The power W_l, propagating in the $l-th$ mode is

$$W_l = \int \left\langle [\vec{E}\times\vec{H}] \right\rangle_t dxdy = A_l^2 N_l^2, \tag{6}$$

$$N_l^2 = \epsilon_0 \frac{c n_l}{2}\int |\vec{\psi}_l(x,y)| dxdy, \tag{7}$$

$$n_l = \frac{c k_l}{\omega}, \tag{8}$$

where n_l is the effective mode refractive index. For the following analysis we also introduce the complex amplitude F of the electric field inside each mode, such that

$$A_l \cos(k_l z - \omega t + \phi_l) = \frac{F_l}{2N_l}e^{i(k_l z - \omega t)} + \frac{F_l^*}{2N_l}e^{-i(k_l z - \omega t)}. \tag{9}$$

In the absence of nonlinearity waveguide modes at different frequencies propagate independently and their amplitudes F are constant over propagation. The idea of the coupled mode theory is to derive the evolution of mode amplitudes $F(z)$ due to nonlinearity.

1. Assume that modes with 3 frequencies $\omega_a > \omega_p > \omega_s$ are propagating along the waveguide, with $\omega_a - \omega_p = \omega_p - \omega_s = \Delta\omega$. The index p denotes the "pump" mode, s and a stand for Stokes- and anti-Stokes modes. Plug in this ansatz,

$$\vec{E}(x,y,z,t) = \frac{F_a}{2N_a}\vec{\psi}_a(x,y)e^{i(k_a z - \omega_a t)} + \frac{F_p}{2N_p}\vec{\psi}_p(x,y)e^{i(k_p z - \omega_p t)} + \frac{F_s}{2N_s}\vec{\psi}_s(x,y)e^{i(k_s z - \omega_s t)} + c.c.$$

in the Eq. (4), collect terms with the frequencies $\omega_{a,p,s}$ and neglect the terms small due to the slow evolution of $F(z)$. Assume also the Stokes and anti-Stokes fields to be much weaker than the pump field for simplicity.

2. Then reduce the resulting equation to the set of 1-st order ODE's for the evolution of $F_{a,p,s}(z)$. These are the coupled-mode equations.

3. Under the approximation that the depletion of pump field is negligible ($F_p(z) = \text{const}$), solve the coupled-mode equations for the evolution of $F_{a,s}(z)$.

4. Show that in such case the magnitudes of Stokes and anti-Stokes fields are exponentially increasing $\propto e^{gz}$ and find the gain coefficient g. This effect is called parametric amplification.

5. For what mismatch in the propagation constants $\Delta\tilde{k} = 2\tilde{k}_p - \tilde{k}_s - \tilde{k}_a$ ($\tilde{k}$ includes contribution due to intensity-dependent refractive index) does the parametric amplification occur? This defines the bandwidth of parametric gain.

[2] One subtle point in the derivation of this equation is that it assumes $\vec{\nabla}\cdot\vec{P}_{\rm NL} = 0$. In most of the practical cases $\vec{\nabla}\cdot\vec{P}_{\rm NL}$ can be neglected, especially in the present case when the coupled-mode equations are valid.

12.5 Nonlinearity enhanecement in ENZ materials

A special class of materials with effective permittivity and/or permeability near zero is called epsilon-near-zero (ENZ) materials. One of the interesting features of such materials is the significant enhancement of intensity-dependent refractive index n_2 ($n = n_0 + n_2 I$) at the wavelength with $\epsilon_0 \to 0$ (see M. Zahirul Alam *et al*, Science **352** (6287)). Show the possibility of n_2 enhancement in ENZ materials by deriving the connection between n_2 and Kerr-nonlinear coefficient $\chi^{(3)}$. Start with calculating χ_{eff} for total polarization of ENZ material ($P^{TOT}(t) = \epsilon_0 \chi_{\text{eff}} E(t)$) including only linear ($P_L(t) = \epsilon_0 \chi^{(1)} E_0 cos(\omega t)$) and third-order nonlinear components ($P_{NL} = \epsilon_0 \chi^{(3)} (E_0 cos(\omega t))^3$). Using the universal expression for the effective susceptibility: $n^2 = 1 + \chi^2_{\text{eff}}$ - derive the final connection between n_2 and $\chi^{(3)}$. Explain what happens when $n_0 \to 0$